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A Combinatorial Interpretation of the Generalized Fibonacci Numbers

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In this paper the generalized Fibonacci numbers of order k are combinatorially interpreted, in the context of the theory of linear species of Joyal, as the linear species of k-filtering partitions. © 1997 Academic Press

1. PARTITIONS AND FIBONACCI NUMBERS

The purpose of this article is to give a combinatorial interpretation of the generalized Fibonacci numbers of order k, i.e., of those numbers $F_n^{(k)}$ defined by the recurrence relation

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \cdots + F_{n+1}^{(k)} + F_n^{(k)} \tag{1}$$

and (as in [5], [8]) by the initial conditions

$$F_0^{(k)} = 0,\ F_2^{(k)} = 0,\ \ldots,\ F_{k-2}^{(k)} = 0,\ F_{k-1}^{(k)} = 1. \tag{2}$$

As is well known, numbers can have more than one combinatorial interpretation and clearly this is also true for generalized Fibonacci numbers. Our interpretation of these numbers is the one that naturally arises in the context of the theory of linear Joyal species.

1.1. *Linear Species*

We briefly recall the concept of linear species, pointing out those aspects that will be used in the following and referring the reader to [4] for further details.

A *linearly ordered set*, or a *linear order*, $\langle L, \leq \rangle$ is a set L together with a linear order relation $\leq$ (i.e., a reflexive, antisymmetric, and transitive

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relation in which all elements are comparable). As usual, we shall write L for $\langle L, \leq \rangle$ and $|L|$ for its cardinality. When L is finite we write $L = [x_1, x_2, \ldots, x_n]$.

A k = *interval* of L is a subset of the form

$$[x; y] = \{u \in L: x \leq u \leq y\}$$

with exactly k elements.

An *order function* $\mu: H \to L$ between two linearly ordered sets $\langle H, \leq_H \rangle$ and $\langle L, \leq_L \rangle$ is a function such that $\mu(x) \leq_L \mu(y)$ whenever $x \leq_H y$. Note that there exists at most one *order bijection* between two finite linear orders H and L and that there exists exactly one when H and L have the same cardinality.

Let **Lin** be the category of finite linearly ordered sets and order bijections, and let **Set** be the category of finite sets and functions.

A *linear species of combinatorial structures* is a functor S: **Lin** $\to$ **Set**. As in [4], $\mathsf{S}[L]$ is the set of all the *structures* of species S on the finite linear order L, and $\mathsf{S}[b]: \mathsf{S}[H] \to \mathsf{S}[L]$ is the *transport of structures* along the order bijection $b: H \to L$.

The *cardinality* of a linear species S is the *formal geometric series*

$$|\mathsf{S}| = \sum_{n \geq 0} |\mathsf{S}[\underline{n}]| t^n,$$

where $\underline{0} := \varnothing$ and $\underline{n} := [1, 2, \ldots, n]$ for $n > 0$.

Two species S and T are *isomorphic* when there exists a natural equivalence between the functors S and T. In this case $|\mathsf{S}| = |\mathsf{T}|$.

The *sum* of two linear species S and T is the linear species $\mathsf{S} + \mathsf{T}$ defined by

$$(\mathsf{S} + \mathsf{T})[L] := \mathsf{S}[L] + \mathsf{T}[L],$$

where the sum on the right is the disjoint union of sets.

The *product* of two linear species S and T is the linear species $\mathsf{S} \cdot \mathsf{T}$ defined by

$$(\mathsf{S} \cdot \mathsf{T})[L] := \sum_{L_1 + L_2 = L} \mathsf{S}[L_1] \times \mathsf{T}[L_2].$$

A *linear partition* π of a finite linear order L is a family of nonempty disjoint intervals, called *blocks*, whose union is L. π is itself a finite linear order.

Let S and T be two linear species, with $\mathsf{T}[\varnothing] = \varnothing$. The *composition*, or *substitution*, $\mathsf{S} \circ \mathsf{T}$ is the species defined in such a manner that giving a structure of species $\mathsf{S} \circ \mathsf{T}$ on a finite linear order L means giving a partition π of L, a structure of species T on each block of π, and a structure of species S on π.

The sum, the product, and the composition of species are all preserved by passing to the cardinalities:

$$|S+T| = |S| + |T|, \qquad |S \cdot T| = |S| \cdot |T|, \qquad |S \circ T| = |S| \circ |T|.$$

To develop our interpretation we need an operator acting on the linear species as the shift operator acts on the sequences of natural numbers. We now introduce such an operator.

Let L be a finite linear order. An *augmentation* of L is a linear order obtained by adding to L a new element. Clearly, we can have more than one augmentation of L even for the same new element.

We say that $\alpha\colon \mathbf{Lin} \to \mathbf{Lin}$ is an *augmentation functor* if $\alpha(L)$ is an augmentation of L, for every $L \in \mathbf{Lin}$.

Let α be an augmentation functor and let S be a linear species. The composition

$$\mathbf{Lin} \xrightarrow{\alpha} \mathbf{Lin} \xrightarrow{S} \mathbf{Set}$$

is also a functor, i.e. a linear species. Therefore we can define the operator $\mathbf{R}_\alpha$ by setting

$$\mathbf{R}_\alpha S := \alpha S,$$

so that

$$(\mathbf{R}_\alpha S)[L] = S[\alpha(L)], \qquad (\mathbf{R}_\alpha S)[b] = S[\alpha(b)]$$

for every finite linear order H and L, and for every order bijection $b\colon H \to L$.

The operator $\mathbf{R}_\alpha$ does not depend, up to natural isomorphism, on the functor α. In other words, if α_1 and α_2 are two augmentation functors, then $\mathbf{R}_{\alpha 1} S$ and $\mathbf{R}_{\alpha 2} S$ are naturally equivalent, for every linear species S.

Because $\alpha_1(L)$ and $\alpha_2(L)$ have the same cardinality, for every $L \in \mathbf{Lin}$, there exists exactly one order bijection

$$b_L\colon \alpha_1(L) \to \alpha_2(L).$$

Therefore we can define a natural equivalence η between $\mathbf{R}_{\alpha_1} S$ and $\mathbf{R}_{\alpha_2} S$ by setting $\eta_L := S[b_L]$, for every $L \in \mathbf{Lin}$.

From now on we shall write $\mathbf{R}$ without any reference to the underlying augmentation functor. So to give a structure of species $\mathbf{R}\,S$ on a finite linear order L means to give an augmentation of L and then a structure of species S on that augmentation.

For our purposes we shall only consider the augmentation functors $1+(-)$ and $(-)+1$ that act by adding a new left element and a new right element respectively. In other words, if $L=[x_1,\ldots,x_n]$ and $x\notin L$, then

$$1+L := [x, x_1,\ldots,x_n], \qquad L+1 := [x_1,\ldots,x_n,x].$$

In general, we shall write $h+L$ and $L+h$, with $h\in\mathbf{N}$, for the linear orders obtained from L, adding h new left elements and h new right elements respectively. Thus, for the hth iterate of the operator $\mathbf{R}$ of a linear species S, we shall have

$$(\mathbf{R}^h\mathsf{S})[L]=\mathsf{S}[h+L] \qquad \text{or} \qquad (\mathbf{R}^h\mathsf{S})[L]=\mathsf{S}[L+h].$$

Now we have to see how $\mathbf{R}$ acts on the formal geometric series. If $S(t)=\sum s_n t^n$ is the cardinality of S, then the cardinality of $\mathbf{R}\,\mathsf{S}$ is the series

$$\mathbf{R}S(t) = \sum_{n\geq 0} s_{n+1}t^n = \frac{S(t)-S(0)}{t}.$$

So $\mathbf{R}S(t)$ is the incremental ratio (in 0) of the series $S(t)$. For this reason we call $\mathbf{R}\,\mathsf{S}$ the *incremental ratio* of the linear species S.

A *linear species in h-sorts*, with $h\in\mathbf{N}$, is a functor $\mathsf{S}:\mathbf{Lin}^h\to\mathbf{Set}$, and the *incremental ratio with respect to the ith sort of* S is defined by

$$(\mathbf{R}_i\,\mathsf{S})[L_1,\ldots,L_i,\ldots,L_h] := \mathsf{S}[L_1,\ldots,\alpha(L_i),\ldots,L_h],$$

where α is an augmentation functor.

We now present some important linear species which will play a central role in our interpretation.

The *geometric species*, or *uniform species*, G is the species defined by $\mathsf{G}[L]:=\{*\}$, for all finite linear orders L. Note that, since G is the singleton on every L, we have $\mathbf{R}\,\mathsf{G}=\mathsf{G}$. The cardinality of G is the geometric series

$$G(t) := \sum_{n\geq 0} t^n = \frac{1}{1-t}.$$

The linear species $\mathsf{G}-1$ is defined by

$$(\mathsf{G}-1)[L] := \begin{cases} \varnothing & L=\varnothing \\ \{*\} & L\neq\varnothing, \end{cases}$$

for every $L \in \mathbf{Lin}$, and its cardinality is the series

$$(G-1)(t) = \frac{1}{1-t} - 1 = \frac{t}{1-t}.$$

The *h-geometric species* G_h is defined as the singleton on every linear order L of cardinality h and as the empty set in all other cases:

$$\mathsf{G}_h[L] := \begin{cases} \{*\} & |L| = h \\ \varnothing & |L| \neq h. \end{cases}$$

So $\mathbf{R}\,\mathsf{G}_{h+1} = \mathsf{G}_h$, and the cardinality of G_h is the series $G_h(t) = t^h$.

1.2. *The Linear Species of Linear Partitions*

A *linear partition* of $L \in \mathbf{Lin}$ is a family π of disjoint nonempty intervals whose union is L. We shall say that B is an *h-block* of π when B has cardinality h.

Let P be the linear species of the linear partitions. To give a linear partition of a finite linear order L means to give a partition π of L, a structure of a nonempty linear order on each block of π, and a geometric structure on π. Therefore we have the isomorphism

$$\mathsf{P} = \mathsf{G} \circ (\mathsf{G} - 1). \tag{3}$$

Then, passing to cardinalities and setting $|\mathsf{P}| := P(t) = \sum p_n t^n$, the relation (3) becomes

$$P(t) = \frac{1}{1-t} \circ \frac{t}{1-t} = \frac{1}{1 - \dfrac{t}{1-t}}.$$

Expanding the obtained series, we have

$$p_n = (2 - [n \neq 0])2^{n-1}, \tag{4}$$

where $[n \neq 0] = 0$ if $n = 0$ and $[n \neq 0] = 1$ if $n \neq 0$.

1.3. *The Linear Species of the k-Filtering Partitions*

A *filtering partition of order k*, or a *k-filtering partition*, of $L \in \mathbf{Lin}$ is a linear partition of L in which each block has at most k elements.

For example, the 2-filtering partitions of $L = [1,2,3,4]$ are

$$\{[1],[2],[3],[4]\}, \{[1,2],[3],[4]\}, \{[1],[2,3],[4]\},$$
$$\{[1],[2],[3,4]\}, \{[1,2],[3,4]\}.$$

Let $F^{[k]}$ be the linear species of the k-filtering partitions, and let $f_n^{(k)}$ be the cardinality of $F^{[k]}[L]$, when $|L| = n$.

To give a k-filtering partition of a finite linear order L means to give a partition π of L, a structure of linear order of cardinality at most k on each block of π, and a geometric structure on π. Therefore

$$F^{[k]} = G \circ (G_1 + G_2 + \cdots + G_n), \tag{5}$$

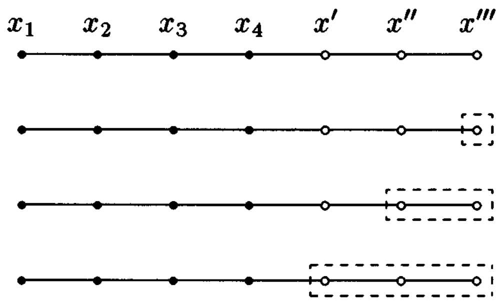
and, passing to cardinalities,

$$\sum_{n \geq 0} f_n^{(k)} t^n = \frac{1}{1-t} \circ (t + t^2 + \cdots + t^k) = \frac{1}{1 - (t + t^2 + \cdots + t^k)}. \tag{6}$$

Let us now consider the species $\mathbf{R}^k F^{[k]}$, with $k \geq 1$. For every finite linear order L, we have, by definition,

$$\mathbf{R}^k F^{[k]}[L] = F^{[k]}[L + k].$$

The set $F^{[k]}[L + k]$ can be partitioned according to the fact that the last block of a k-filtering partition π of $L + k$ has cardinality 1, or $2, \ldots,$ or k. For example, if $|L| = 4$ and $k = 3$, we have the following cases:



Obviously there is a bijection between the set of all k-filtering partitions of $L + k$ with an h-block $(1 \leq h \leq k)$ as the last block and the set of all k-filtering partitions of $L + (k - h)$.

Therefore we have the isomorphism

$$\mathbf{R}^k F^{[k]} = \mathbf{R}^{k-1} F^{[k]} + \mathbf{R}^{k-2} F^{[k]} + \cdots + \mathbf{R} F^{[k]} + F^{[k]} \tag{7}$$

and, passing to cardinalities, the recurrence relation

$$f_{n+k}^{(k)} = f_{n+k-1}^{(k)} + f_{n+k-2}^{(k)} + \cdots + f_{n+1}^{(k)} + f_n^{(k)} \tag{8}$$

which is exactly the relation (1) by means of which the generalized Fibonacci numbers of order k are defined.

To obtain the initial conditions it suffices to observe that a k-filtering partition of a linear-order L of cardinality less than (or equal to) k is an arbitrary linear partition of L. Therefore, due to (4) in the preceeding subsection, we have

$$f_0^{(k)} = 1,\ f_1^{(k)} = 1,\ f_2^{(k)} = 2, \ldots, f_{k-2}^{(k)} = 2^{k-3},\ f_{k-1}^{(k)} = 2^{k-2}. \tag{9}$$

So our initial conditions are different from the usual ones given by (2). Yet our Fibonacci numbers differ from the usual ones only be a shifting; more precisely, for every $k \geq 1$, we have

$$f_n^{(k)} = F_{n+k-1}^{(k)}.$$

1.4. *The Linear Species of the k-Filtering Partitions with h Blocks*

Let $\mathsf{F}_h^{[k]}$ be the linear species of the k-filtering partitions with h blocks and let $f_{h,n}^{(k)}$ be the cardinality of $|\mathsf{F}_h^{[k]}[L]|$, when $|L| = n$.

To give a structure of species $\mathsf{F}_h^{[k]}$ on a finite linear order L means to give a partition π of L, a structure of linear order with at most k elements on each block of π, and a structure of a linear order with h elements on π. Therefore we have

$$\mathsf{F}_h^{[k]} = \mathsf{G}_h \circ (\mathsf{G}_1 + \mathsf{G}_2 + \cdots + \mathsf{G}_k). \tag{10}$$

Moreover, the family $\{\mathsf{F}_h^{[k]}\}_{h \in \mathbf{N}}$ being summable, we also have

$$\mathsf{F}^{[k]} = \sum_{h \geq 0} \mathsf{F}_h^{[k]} \tag{11}$$

or

$$\mathsf{G} \circ (\mathsf{G}_1 + \mathsf{G}_2 + \cdots + \mathsf{G}_k) = \sum_{h \geq 0} \mathsf{G}_h \circ (\mathsf{G}_1 + \mathsf{G}_2 + \cdots + \mathsf{G}_k).$$

Passing to cardinalities, from (10) we obtain

$$\sum_{n \geq 0} f_{h,n}^{(k)} t^n = t^h \circ (t + t^2 + \cdots + t^k) = (t + t^2 + \cdots + t^k)^h, \tag{13}$$

whereas from (11) we obtain again the series (6).

Expanding the series at the right-hand side of (13), it is easy to find

$$f_{h,n}^{(k)} = \sum_{\substack{r_1,\ldots,r_k \geq 0 \\ r_1 + \cdots + r_k = h \\ 1r_1 + \cdots + kr_k = n}} \binom{h}{r_1,\ldots,r_k}, \tag{14}$$

and then, summing up to h,

$$f_n^{(k)} = \sum_{h \geq 0} f_{h.n}^{(k)} = \sum_{\substack{r_1,\ldots,r_k \geq 0 \\ 1r_1 + \cdots + kr_k = n}} \binom{r_1 + \cdots + r_k}{r_1,\ldots,r_k}. \tag{15}$$

Let us consider the species $\mathbf{R}^k F_{h+1}^{[k]}$. As the set $F_{h+1}^{[k]}[L+k]$ can be partitioned according to the fact that the last block of a k-filtering partition of $L+k$ has cardinality at most k, we have

$$\mathbf{R}^k F_{h+1}^{[k]} = \mathbf{R}^{k-1} F_h^{[k]} + \mathbf{R}^{k-2} F_h^{[k]} + \cdots + \mathbf{R} F_h^{[k]} + \mathbf{R} F_h^{[k]}. \tag{16}$$

Passing to cardinalities, we obtain the recurrence relation

$$f_{h+1,n+k}^{(k)} = f_{h,n+k-1}^{(k)} + f_{h,n+k-2}^{(k)} + \cdots + f_{h,n+1}^{(k)} + f_{h,n}^{(k)} \tag{17}$$

with the initial conditions

$$\begin{aligned} f_{0,n}^{(k)} &= [n=0] \text{ and } f_{h,0}^{(k)} = [h=0], \\ f_{h,j}^{(k)} &= 2^{j-1}[h \leq j],\ j = 1,\ldots,k-1. \end{aligned} \tag{18}$$

2. APPLICATIONS

2.1. *First Application*

Let $H_2^{[k]}$ be the linear species in two sorts defined, for every $H, L \in \mathbf{Lin}$, by

$$H_2^{[k]}[H,L] := F^{[k]}[H+L].$$

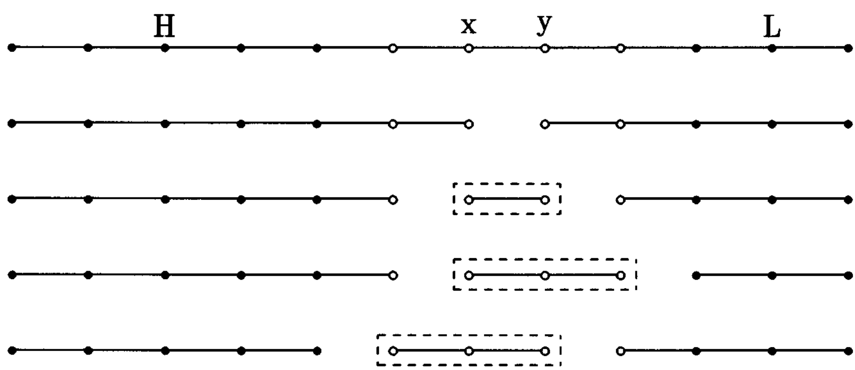
Let us consider, for $k \geq 1$, the species $\mathbf{R}_1^{k-1}\mathbf{R}_2^{k-1}H_2^{[k]}$; for $H, L \in \mathbf{Lin}$ we have

$$\begin{aligned} \mathbf{R}_1^{k-1}\mathbf{R}_2^{k-1}H_2^{[k]}[H,L] &= H_2^{[k]}[H+(k-1),(k-1)+L] \\ &= F^{[k]}[H+(k-1)+(k-1)+L]. \end{aligned} \tag{19}$$

Let x and y be the last element of $H + (k-1)$ and the first element of $(k-1) + L$ respectively. The set (19) can be partitioned according to the fact that in a k-filtering partition π of $H + (k-1) + (k-1) + L$, x and y belong either to different blocks or to the same block.

In the first case, π decomposes in an arbitrary k-filtering partition of $H + (k-1)$ and in an arbitrary k-filtering partition of $(k-1) + L$. In the second case, x and y belong to an interval with at most k elements; so this interval contains the last l elements of $H + (k-1)$ and the first r elements of $(k-1) + L$, where $l \geq 1$, $r \geq 1$, and $l + r \leq k$. Thus π decomposes in an arbitrary k-filtering partition of $H + (k-1-l)$ and in an arbitrary k-filtering partition of $(k-1-r) + L$.

For example, if $|H| = 5$, $|L| = 3$, and $k = 3$, there are the following cases:



This observation immediately yields the following isomorphism:

$$\mathbf{R}_1^{k-1}\mathbf{R}_2^{k-1}H_2^{[k]}[H, L] = F^{[k]}[H + (k-1)] \times F^{[k]}[(k-1) + L] + \sum_{\substack{l, r \geq 1 \\ l + r \leq k}} F^{[k]}[H + (k-1-l)] \times F^{[k]}[(k-1-r) + L]. \qquad (20)$$

Therefore, passing to cardinalities, we have the identity

$$f^{(k)}_{m+n+2k-2} = f^{(k)}_{m+k-1} f^{(k)}_{n+k-1} + \sum_{\substack{l, r \geq 0 \\ l + r \leq k}} f^{(k)}_{m+k-1-l} f^{(k)}_{n+k-1-r}, \qquad (21)$$

and, for $m = n$, the identity

$$f^{(k)}_{2n+2k-2} = \left(f^{(k)}_{n+k-1}\right)^2 + \sum_{\substack{l, r \geq 1 \\ l + r \leq k}} f^{(k)}_{n+k-1-l} f^{(k)}_{n+k-1-r}. \qquad (22)$$

There follow the instances of the preceding formulas for $k = 2, 3, 4$:

$$\begin{aligned}
f_{m+n+2} &= f_{m+1} f_{n+1} + f_m f_n \\
f_{2n+2} &= f_{n+1}^2 + f_n^2 \\
g_{m+n+4} &= g_{m+2} g_{n+2} + g_{m+1} g_{n+1} + g_{m+1} g_n + g_m g_{n+1} \\
g_{2n+4} &= g_{n+2}^2 + g_{n+1}^2 + 2 g_{n+1} g_n \\
h_{m+n+6} &= h_{m+3} h_{n+3} + h_{m+2} h_{n+2} + h_{m+1} h_{n+1} \\
&\qquad + h_{m+2} h_{n+1} + h_{m+1} h_{n+2} + h_{m+2} h_n + h_m h_{n+1} \\
h_{2n+6} &= h_{n+3}^2 + h_{n+2}^2 + h_{n+1}^2 + 2(h_{n+1} + h_n) h_{n+2},
\end{aligned}$$

where $f_n := f_n^{(2)}$, $g_n := f_n^{(3)}$, and $h_n := f_n^{(4)}$.

Let us now consider the species $\mathsf{H}_h^{[k]}$ in h sorts defined by

$$\mathsf{H}_h^{[k]}[L_1, \dots, L_h] := \mathsf{F}^{[k]}[L_1 + \cdots + L_h]$$

for all $L_1, \dots, L_h \in \mathbf{Lin}$, and, for $h, k \geq 1$, the species

$$\mathbf{R}_1^{k-1} \mathbf{R}_2^{2(k-1)} \cdots \mathbf{R}_{h-1}^{2(k-1)} \mathbf{R}_h^{k-1} \mathsf{H}_h^{[k]}.$$

We have

$$\mathbf{R}_1^{k-1} \mathbf{R}_2^{2(k-1)} \cdots \mathbf{R}_{h-1}^{2(k-1)} \mathbf{R}_h^{k-1} \mathsf{H}_h^{[k]}[L_1, \dots, L_h] \\ = \mathsf{H}_h^{[k]}[H_1, H_2, \dots, H_{h-1}, H_h], \quad (23)$$

where

$$\begin{aligned}
H_1 &:= L_1 + (k - 1 - l_1), \\
H_2 &:= (k - 1 - r_1) + L_2 + (k - 1 - l_2), \\
&\ \vdots \\
H_{h-1} &:= (k - 1 - r_{h-2}) + L_{h-1} + (k - 1 - l_{h-1}), \\
H_h &:= (k - 1 - r_{h-1}) + L_h.
\end{aligned}$$

Let x_i and y_i be the last element of H_i and the first element of H_{i+1} respectively, for $i = 1, 2, \dots, h - 1$. The set (23) can be partitioned according to the fact that x_i and y_i either belong to different blocks or belong to the same block, for every $i = 1, 2, \dots, h-1$. So we have

$$\begin{aligned}
&\mathbf{R}_1^{k-1}\mathbf{R}_2^{2(k-1)} \cdots \mathbf{R}_{h-1}^{2(k-1)}\mathbf{R}_h^{k-1} H_h^{[k]}[L_1, \dots, L_h] \\
&\quad = \sum_{\chi : \underline{h-1} \to \{0,1\}} \sum_{\substack{l_i, r_i \geq \chi(i) \\ l_i + r_i \leq k\chi(i) \\ i = 1, \dots, h-1}} \\
&\qquad \times \prod_{j=1}^{h} F^{[k]}[H_j] \qquad (24)
\end{aligned}$$

where χ is the function by means of which we distinguish the case in which x_i and y_i belong to different blocks ($\chi(i) = 0$)) from the case in which x_i and y_i are in the same block ($\chi(i) = 1$)).

Finally, passing to cardinalities, we have

$$f^{(k)}_{n_1 + \cdots + n_h + 2(h-1)(k-1)} = \sum_{\chi : \underline{h-1} \to \{0,1\}} \sum_{\substack{l_i, r_i \geq \chi(i) \\ l_i + r_i \leq k\chi(i) \\ i = 1, \dots, h-1}} f^{(k)}_{s_1} f^{(k)}_{s_2} \cdots f^{(k)}_{s_{h-1}} f^{(k)}_{s_h}, \quad (25)$$

where

$$\begin{aligned}
s_1 &:= n_1 + k - 1 - l_1, \\
s_2 &:= n_2 + 2k - 2 - r_1 - l_2, \\
&\ \ \vdots \\
s_{h-1} &:= n_{h-1} + 2k - 2 - r_{h-2} - l_{h-1}, \\
s_h &:= n_h + k - 1 - r_{h-1}.
\end{aligned}$$

There follow the formulas obtained from (25) when $n_1 = \cdots = n_h = n$ and $(h, k) = (3, 2), (4, 2), (3, 3)$:

$$\begin{aligned}
f_{3n+4} &= f_{n+1}^3 + 3f_{n+1}^2 f_n + f_n^3 \\
f_{4n+6} &= 2f_{n+1}^4 + 4f_{n+1}^3 f_n + 6f_{n+1}^2 f_n^2 + f_n^4 \\
g_{3n+8} &= 2g_{n+2}^3 + 2g_{n+1}^3 + (g_{n+2} + g_n)\left(6g_{n+2}g_{n+1} + 3g_{n+2}g_n + 3g_{n+1}^2\right).
\end{aligned}$$

2.2. *Second Application*

Let us consider the species $F^{[k+m]}$, with $m \geq 1$. Given $\pi \in F^{[k+m]}[L]$, we can partition π according to the fact that any block has cardinality $\leq k$ or $> k$. Thus we can give a structure of species $F^{[k+m]}$ in the following way. First, we give two distinct intervals L_1 and L_2 whose union is L, and a structure of species $F^{[k]}$ on L_2. Then we give a partition π of L_1, for each block B of π two disjoint intervals B_1 and B_2 whose union is B, a structure of species $\mathsf{F}^{[k]}$ on B_1, and a structure of linear order with cardinality between $k+1$ and $k+m$ on B_2. Therefore we have the isomorphism

$$\mathsf{F}^{[k+m]} = \left[G \circ \left(\mathsf{F}^{[k]} \cdot (\mathsf{G}_{k+1} + \cdots + \mathsf{G}_{k+m})\right)\right] \cdot \mathsf{F}^{[k]} \tag{26}$$

and, passing to cardinalities,

$$\frac{1}{1-(t+\cdots+t^{k+m})} = \frac{1}{1-\dfrac{t^{k+1}+\cdots+t^{k+m}}{1-(t+\cdots+t^{k})}} \cdot \frac{1}{1-(t+\cdots+t^{k})}. \tag{27}$$

On the other hand, $\mathsf{F}^{[k]}[L]$ can be partitioned according to the number of blocks of a k-filtering partition of L having cardinality between $k+1$ and $k+m$. Let $\mathsf{C}_r^{[k+m]}$ the species of the k-filtering partition with r such blocks.

Now a partition $\pi \in \mathsf{C}_r^{[k+m]}$ determines a $(2r+1)$-tuple

$$(H_1, C_1, H_2, C_2, \ldots, H_r, C_r, H_{r+1})$$

of disjoint intervals of L whose union is L, where the C_j are the r intervals with $k+1 \leq |C_j| \leq k+m$, and the H_j are arbitrary intervals (eventually empty). Let $\mathsf{P}_r^{k,m}[L]$ be the set of such $(2r+1)$-tuples.

Clearly, we have

$$\begin{aligned}
\mathsf{F}^{[k+m]}[L] &= \sum_{r \geq 0} \mathsf{C}_r^{[k+m]}[L] \\
&= \sum_{r \geq 0} \sum_{\substack{\pi \in \mathsf{P}_r^{k,m}[L] \\ \pi = (H_1, C_1, \ldots, H_{r+1})}} \\
&\qquad \times \prod_{j=1}^{r+1} \mathsf{F}^{[k]}[H_j]
\end{aligned} \tag{28}$$

Thus, passing to cardinalities, we obtain

$$f_n^{(k+m)} = \sum_{r \geq 0} \sum_{\substack{\pi \in \mathsf{P}_r^{k,m}[L] \\ \pi = (H_1, C_1, \ldots, H_{r+1})}} f_{|H_1|}^{(k)} \cdots f_{|H_{r+1}|}^{(k)}. \tag{29}$$

It is clear that for every

$$(H_{1,1}, C_1, \ldots, H_{1,r}, C_r, H_{1,r+1})$$

and

$$(H_{2,1}, C_{p(1)}, \ldots, H_{2,r}, C_{p(r)}, H_{2,r+1}),$$

where $H_{1,j} \sim H_{2,j}$ and p is a permutation of $\underline{r}$, we have

$$f^{(k)}_{|H_{1,1}|} \cdots f^{(k)}_{|H_{1,r+1}|} = f^{(k)}_{|H_{2,1}|} \cdots f^{(k)}_{|H_{2,r+1}|}.$$

So, if we have r_j $(k+j)$-blocks, with $j = 1, \ldots, m$, then (29) becomes

$$f_n^{(k+m)} = \sum_{\substack{r_1, \ldots, r_m \geq 0 \\ r := r_1 + \cdots + r_m}} \binom{r}{r_1, \ldots, r_m} g_n(r_1, \ldots, r_m) \tag{30}$$

where

$$g_n(r_1, \ldots, r_m) = \sum_{\substack{h_1, \ldots, h_{r+1} \geq 0 \\ h_1 + \cdots + h_{r+1} = n - r_1(k+1) - \cdots - r_m(k+m)}} f_{h_1}^{(k)} \cdots f_{h_{r+1}}^{(k)}.$$

Obviously this result can also be obtained by formally expanding the series at the right-hand side of (27).

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